Asymptotic behavior of a general class of mixture failure rates

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Abstract

Mixture os increasing failure rate distributions (IFR) can decrease at least in some intervals of time. Usually this property is observed asymptotically as $t \to \infty$, which is due to the fact that a mixture failure rate is 'bent down', as the weakest populations are dying out first. We consider a survival model, generalizing very well known in reliability and survival analysis additive hazards, proportional hazards and accelerated life model. We obtain new explicit asymptotic relations for a general setting and study specific cases. Under reasonable assumptions we prove that asymptotic behavior of the mixture failure rate depends only on the behavior of the mixing distribution in the neighborhood of the left end point of its support and not on the whole mixing distribution.

Keywords: mixture of distributions, decreasing failure rate, increasing failure rate, proportional hazardx model, accelerated life model.

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1 Introduction

Mixtures of decreasing failure rate (DFR) distributions are always DFR (Barlow and Proschan, 1975). On the other hand, mixtures of increasing failure rate distributions (IFR) can decrease at least in some intervals of time, which means that the IFR class of distributions is not closed under the operation of mixing (Lynch, 1999). As IFR distributions usually model lifetimes governed by aging processes, it means that the operation of mixing can change the pattern of aging, e.g., from positive aging (IFR) to the negative aging (DFR). These important facts should be taken into account in applications.

One can hardly find homogeneous populations in real life and mixtures of distributions usually present an effective tool for modelling heterogeneity. A natural specific approach for this modelling exploits a notion of a non-negative random unobserved parameter (frailty) Z introduced by Vaupel *et al* (1979) in a demographic context. This, in fact, leads to considering a random failure rate $\lambda(t, Z)$. As the failure rate is a conditional characteristic, the 'ordinary' expectation $E[\lambda(t, Z)]$ with respect to Z does not define a mixture failure rate $\lambda_m(t)$ and a proper conditioning should be performed (Finkelstein, 2004).

A convincing 'experiment', showing the deceleration in the observed failure rate is performed by nature. It is well-known that the human mortality follows the Gompertz lifetime distribution with exponentially increasing mortality rate. Assume that heterogeneity can be described by the proportional Gamma-frailty model:

$$\lambda(t, Z) = Z\alpha e^{\beta t},$$

where α and β are positive constants, defining a baseline mortality rate. Due to the computational simplicity, the Gamma-frailty model is practically the only one used in applications so far.

It can be shown (see, e.g., equation (29) in the current paper) that the mixture failure rate $\lambda_m(t)$ in this case is monotone in $[0, \infty]$ and asymptotically tends to a constant as $t \to \infty$. It is monotonically increasing, however, for the real values of parameters of this model. This fact explains recently observed deceleration in human mortality for oldest old (human mortality plateau, as in Thatcher (1999)). A similar result is experimentally obtained for a large cohort of medflies by Carey *et al* . On the other hand, in engineering applications the operation of mixing can result even in a more dramatic effect: the mixture failure rate is increasing in $[0, t_m), t_m > 0$ and decreasing asymptotically to 0 in $[t_m, \infty)$, which, e.g., was experimentally observed in Finkelstein (2005) for the heterogeneous sample of miniature light bulbs. This fact is easily explained theoretically via the gamma frailty model with a baseline failure rate increasing, in accordance with a Weibull law, as a power function (Gupta and Gupta, 1996 ; Finkelstein and Esaulova, 2001).

In Block *et al* (2003) it was proved that if the failure rate of each subpopulation converges to a constant and this convergence is uniform, then the mixture failure rate converges to the failure rate of the strongest subpopulation: the weakest subpopulations are dying out first. (For convenience from now on we shall use where appropriate the term "population" instead of "subpopulation") These authors generalize a case of constant in time failure rates of populations, considered by Clarotti and Spizzichino (1990) and present a further development of Block *et al* (1993) (see also Lynn and Singpurwalla, 1997 ; Gurland and Sethuraman, 1995). In Block and Joe (1997) the following asymptotic result, which also addresses the issue of ultimate monotonicity, was obtained: let z_0 be a realization of a frailty Z, which corresponds to the strongest population. If $\lambda(t, z)/\lambda(t, z_0)$ uniformly decreases as $t \to \infty$, then $\lambda_m(t)/\lambda(t, z_0)$ also decreases. If, in addition, $\lim_{t\to\infty} \lambda(t, z_0)$ exists, then this quotient decreases to 1.

The recent paper of Li (2005) generalizes the results of Block *et al* (2003a), using the similar analytical tools and approaches. Instead of assuming that each individual failure rate has a limit, the author assumes that there exists an asymptotic baseline function $\lambda(t)$ such that the ratio of each individual failure rate with the asymptotic baseline function $\lambda(t, z)/\lambda(t)$ has a limit. He shows that under certain conditions the ratio of the mixture failure rate with the asymptotic baseline function has a limit. As in Block *et al* (2003) it is shown that this limit is the corresponding essential infimum. Again, the stringent condition of the uniform convergence of $\lambda(t, z)/\lambda(t)$ to some $a(z)$ is imposed. Therefore this paper combines the analytical reasoning of Block *et al* (2003a) with the 'ratio approach' of Block and Joe (1997).

The models in the foregoing papers are, in fact, generalized proportional hazards models. They are based on asymptotic equivalence $\lambda(t, z) \sim \lambda(t)a(z)$ in the sense of the uniform convergence of the ratio to 1, which is, as already mentioned, a rather strong assumption. Besides, the strongest population is not always identifiable.

The goal of the current paper is to try to find a balance between the generality of a model and a possibility of obtaining **explicit asymptotic results** for the mixture failure rate $\lambda_m(t)$. We suggest a class of distributions, which hopefully meets this requirement and develop a **new for this kind of applications approach**, related to the ideology of generalized convolutions, e.g., Laplace and Fourie transforms and, especially, Mellin convolutions (Bingham *et al*, 1987). For proving our asymptotic results we use a standard technique similar to the one used for obtaining Abelian, Tauberian and Mercerian-type theorems, although our theorems are not the direct corollaries of results in this field. In line with this relationship it turns out that asymptotic behavior of mixture failure rates for the suggested class of lifetime distributions depends only on the behavior of the mixing distribution in the neighborhood of $\inf_z\{\pi(z) > 0\}$ and not on the whole mixing pdf $\pi(z)$.

After defining a survival model in Section 2, we formulate our main theorems in Section 3 and consider important for applications examples in Section 4. As the proofs are more technical than we hoped them to be, they are deferred to a special Section 5.

2 The survival model

Let $T \geq 0$ be a lifetime random variable with Cdf $F(t)$ ($\bar{F}(t) = 1 - F(t)$). Assume that $F(t)$ is indexed by a non-negative random variable Z with a pdf $\pi(z)$ and support in $[0, \infty)$:

$$P(T \leq t|Z = z) \equiv P(R \leq t|z) = F(t, z)$$

and that the pdf $f(t, z)$ exists. Therefore the corresponding failure rate for every fixed z is $\lambda(t, z) = f(t, z)/\bar{F}(t, z)$. The support $[a, b], a > 0, b < \infty$ can be also considered. Thus, the mixture Cdf and pdf are defined by

$$F_m(t) = \int_0^\infty F(t, z)\pi(z)dz, \qquad f_m(t) = \int_0^\infty f(t, z)\pi(z)dz,$$

respectively. The mixture failure rate is

$$\lambda_m(t) = \frac{f_m(t)}{\bar{F}_m(t)} = \frac{\int_0^\infty f(t, z)\pi(z)dz}{\int_0^\infty \bar{F}(t, z)\pi(z)dz}, \tag{1}$$

Denote, as usually, the cumulative failure rate by:

$$\Lambda(t,z) = \int_0^t \lambda(u,z)du.$$

We will define a class of lifetime distributions $F(t,z)$ and will study asymptotic behavior of the corresponding mixture failure rate $\lambda_m(t)$. It is more convenient at the start to give this definition in terms of the cumulative failure rate $\Lambda(t,z$, rather than in terms of the failure rate $\lambda(t,z)$. The basic model is given by the following relation:

$$\Lambda(t,z) = A(z\phi(t)) + \psi(t). \tag{2}$$

General assumptions for the model (2):
Natural properties of the cumulative failure rate of the absolutely continuous distribution $F(t,z)$ (for $\forall z \in [0,\infty)$) imply that the functions: $A(s), \phi(t)$ and $\psi(t)$ are differentiable, the right hand side of (2) is non-decreasing in t and tends to infinity as $t \to \infty$ and that $A(z\phi(0)) + \psi(0) = 0$. Therefore, these properties will be assumed throughout the paper, although some of them will not be needed for formal proofs.

An important additional simplifying assumption is that

$$A(s), s \in [0,\infty); \phi(t), t \in [0,\infty)$$

are increasing functions of their arguments, although some generalizations (e.g., for ultimately increasing functions) can be easily performed. Therefore, we will view $1 - e^{-A(z\phi(t))}, z \neq 0$ in this paper as a lifetime Cdf.

It should be noted, that model (2) can be also easily generalized to the form $\Lambda(t,z) = A(g(z)\phi(t)) + \psi(t) + \eta(z)$ for some properly defined functions $g(z)$ and $\eta(z)$. However, we cannot go generalizing further (at least, at this stage) and the multiplicative form of arguments in $A(g(z)\phi(t))$ is important for our method of deriving asymptotic relations. It is also clear that the additive term $\psi(t)$, although important in applications, gives only a slight generalization for further analysis of $\lambda_m(t)$, as (2) can be interpreted in terms of two components in series (or, equivalently, via two competing risks). However, this term will be essential in Section 3, while defining the strongest population.

The failure rate corresponding to the cumulative failure rate $\Lambda(t,z)$ is

$$\lambda(t,z) = z\phi'(t)A'(z\phi(t)) + \psi(t) \tag{3}$$

Now we are able to explain, why we start with the cumulative failure rate and not with the failure rate itself, as often in lifetime modeling. The reason is that one can easily suggest intuitive interpretations for (2), whereas it is certainly not so simple to interpret the failure rate structure in the form (3) without stating that it just follows from the structure of the cumulative failure rate.

Relation (2) defines a rather broad class of survival models which can be used, e.g., for modelling an impact of environment on characteristics of survival. The widely used in reliability, survival analysis and risk analysis proportional hazards (PH), additive hazards (AH) and accelerated life (ALM) models, are the obvious specific cases of our relations (2) or (3):

PH (multiplicative) Model:

Let

$$A(u) \equiv u, \quad \phi(t) = \Lambda(t), \quad \psi(t) = 0.$$

Then

$$\lambda(t, z) = z\lambda(t), \quad \Lambda(t, z) = z\Lambda(t). \tag{4}$$

Accelerated Life Model:

Let

$$A(u) \equiv \Lambda(u), \quad \phi(t) = t, \quad \psi(t) = 0.$$

Then

$$\Lambda(t, z) = \int_0^{tz} \lambda(u) du = \Lambda(tz), \quad \lambda(t, z) = z\lambda(tz). \tag{5}$$

AH Model:

Let

$$A(u) \equiv u, \quad \phi(t) = t, \quad \psi(t) \text{ is increasing}, \psi(0) = 0.$$

Then

$$\lambda(t, z) = z + \psi'(t), \quad \Lambda(t, z) = zt + \psi(t). \tag{6}$$

The functions $\lambda(t)$ and $\phi'(t)$ play the role of baseline failure rates in equations (4), (5) and (6), respectively. Note that in all these models, the functions $\phi(t)$ and $A(s)$ are monotonically increasing.

Asymptotic behavior of mixture failure rates for PH and AH models was studied for some specific mixing distributions, e.g., in Gurland and Sethuraman (1995) and Finkelstein and Esaulova (2001a) . On the other hand, as far as we know, the mixture failure rate for the ALM was considered at a descriptive level only in Anderson and Louis (1995).

3 General results

The next theorem derives an asymptotic formula for the mixture failure rate $\lambda_m(t)$ under rather mild assumptions.

Theorem 1 *Let the cumulative failure rate* $\Lambda(t, z)$ *be given by model (2) and the mixing pdf* $\pi(z)$ *be defined as*

$$\pi(z) = z^{\alpha} \pi_1(z), \tag{7}$$

where $\alpha > -1$ *and* $\pi_1(z)$, $\pi_1(0) \neq 0$, *is a bounded in* $[0, \infty)$ *and continuous at* $z = 0$ *function.*

Assume also that

$$\phi(t) \to \infty \quad as \quad t \to \infty \tag{8}$$

and

$$\int_0^{\infty} e^{-A(s)} s^{\alpha} ds < \infty, \tag{9}$$

where $A(s)$ *is also ultimately increasing.*

Then

$$\lambda_m(t) - \psi'(t) \sim (\alpha + 1) \frac{\phi'(t)}{\phi(t)}. \tag{10}$$

By relation (10) we, as usual, mean asymptotic equivalence and write $a(t) \sim b(t)$ as $t \to \infty$, if $\lim_{t \to \infty} [a(t)/b(t)] = 1$.

It is easy to see that assumption (7) holds for the main lifetime distributions such as Weibull, Gamma, lognormal etc. Assumption (8) states a natural condition for the function $\phi(t)$, which can be often viewed as a scale transformation. Condition (9) means that the Cdf $1 - e^{-A(s)}$ should not be 'too heavy tailed' (as e.g. the Pareto distribution $1 - s^{-\beta}$, for $s \geq 1, \beta - \alpha > 1$) and in our assumptions equivalent to the condition of existence of the moment of order $\alpha + 1$ for this Cdf. Examples of the next section will clearly show that these conditions are not stringent at all and can be easily met in most practical situations.

A crucial feature of this result is that asymptotic behavior of the mixture failure rate depends only (omitting an obvious additive term $\psi(t)$) on the behavior of the mixing distribution in the neighborhood of zero and on the derivative of the logarithm of the scale function $\phi(t)$: $(\log\phi(t))' = \phi'(t)/\phi(t)$. When $\pi(0) \neq 0$ and $\pi(z)$ is bounded in $[0, \infty)$, the result does not depend on the mixing distribution at all , as $\alpha = 0$!

Theorem 1 (as well as later theorems 2 and 4) can be formally generalized to the case when the mixing random variable Z does not necessarily posses an absolutely continuous Cdf in $[0, \infty)$: it is sufficient that it should be absolutely continuous (from the right) at $z = 0$.

We can formulate a more general result, which states a similar dependence on the behavior of the mixing distribution at zero in terms of asymptotic comparison of two mixture failure rates:

If, under some assumptions, the two mixing distributions are equivalent at $z = 0$, then the mixture failure rates are equivalent as $t \to \infty$.

Formally:

Theorem 2 *Let $f(t, z)$ and $\pi(z)$ be the lifetime and mixing pdf's in a general mixing model (2), respectively. Assume that there exists a positive function $\alpha(t)$, which is ultimately decreasing to 0 as $t \to \infty$ and that*

$$\frac{\int_0^{\alpha(t)} f(t, z)\pi(z)dz}{\int_0^{\infty} f(t, z)\pi(z)dz} \to 1. \tag{11}$$

Denote another mixing pdf by $\rho(z)$ and assume that $\rho(z)/\pi(z)$ is bounded in $[0, \infty)$, continuous at 0, and $\lim_{z\to 0} \rho(z)/\pi(z) \neq 0$. Then:

$$\lambda_m^{\pi}(t) \equiv \frac{\int_0^{\infty} f(t, z)\pi(z)dz}{\int_0^{\infty} \bar{F}(t, z)\pi(z)dz} \sim \frac{\int_0^{\infty} f(t, z)\rho(z)dz}{\int_0^{\infty} \bar{F}(t, z)\rho(z)dz} \equiv \lambda_m^{\rho}(t) \tag{12}$$

as $t \to \infty$.

It is worth noting that if $\psi \equiv 0$ and all other conditions of Theorem 1 hold, condition (8) of this theorem guarantees assumption (11).

It is important, that for applying Theorem 2 we do not need a specific form of a survival model. As it will be seen from the proof, $\pi(z)$ and $\rho(z)$ also need not necessarily be probability density functions (local integrability, in fact, is sufficient). The following corollary exploits the latter fact for the case when $\pi(z) \equiv 1$:

Corollary 1 *Let $f(x,t)$ be a lifetime pdf in a general mixing model (2). Assume that there exists a positive function $\alpha(t)$, such that $\alpha(t)$ is ultimately decreasing to zero as $t \to \infty$ and*

$$\frac{\int_0^{\alpha(t)} f(t,z)dz}{\int_0^{\infty} f(t,z)dz} \to 1. \tag{13}$$

Let $\rho(z)$ be positive function bounded in $[0,\infty)$, continuous at zero and $\rho(0) \neq 0$. Then:

$$\frac{\int_0^{\infty} f(t,z)\rho(z)dz}{\int_0^{\infty} \bar{F}(t,z)\rho(z)dz} \sim \frac{\int_0^{\infty} f(t,z)dz}{\int_0^{\infty} \bar{F}(t,z)dz}. \tag{14}$$

as $t \to \infty$.

Theorems 1 and 2 consider the case when the support of a mixing distribution includes 0: $z \in [0,\infty)$. If the support is separated from 0, the situation changes significantly and we can observe a well-known principle that the mixture failure rate tends to the failure rate of the strongest population (Block and Joe, 1992; Block et al, 2003; Finkelstein and Esaulova, 2001a).

Theorem 3 *Let the class of lifetime distributions be defined by equation (2), where $\phi(t) \to \infty$ and $A(s)$ is twice differentiable. Assume that as $s \to \infty$:*

$$\frac{A''(s)}{(A'(s))^2} \to 0 \tag{15}$$

and

$$sA'(s) \to \infty \tag{16}$$

Assume also that $\forall\, b,c > 0$, $b < c$ the quotient $A'(bs)/A'(cs)$ is bounded as $s \to \infty$.

Let the mixing pdf $\pi(z)$ be defined in $[a,\infty)$, $a > 0$, bounded in this interval, continuous at $z = a$, and $\pi(a) \neq 0$.

Then

$$\lambda_m(t) - \psi'(t) \sim a\phi'(t)A'(a\phi(t)). \tag{17}$$

It is clear that conditions (15) and (16) trivially hold for specific multiplicative and additive models of the previous section. We will discuss them within the framework of the accelerated life model later. More generally, these conditions hold, if $A(s)$ belongs to a class of functions of smooth variation (Bingham *et al*, 1987).

Assume additionally that the family of failure rates (3) is ordered in z, at least, ultimately:

$$\lambda(t, z_1) < \lambda(t, z_2), \quad \forall z_1, z_2 \in [z_0, \infty),\ z_1 < z_2, z_0 \geq 0, t \geq 0.$$

Then, as it was mentioned, Theorem 3 can be interpreted via the principle that the mixture failure rate converges to the failure rate of the strongest population. (Note that the right hand side in (17) also can be interpreted in this case as the failure rate of the strongest population for a survival model, defined by a random variable with the Cdf $1 - e^{-A(z\phi(t))}$). An interesting question arises: whether this principle is a 'universal law', or a consequence of sufficient assumptions of Theorem 3? Theorem 1 gives us an idea for creating counter-examples:

Example 1 Assume that all conditions of Theorem 1 hold and additionally: $A'(s)$ is increasing in $[0, \infty)$. Then an ordering of failure rates in the family (3) with respect to z (for each fixed $t > 0$) holds resulting **formally** in the strongest population defined as $\lambda(t, 0) = \phi'(t)$. Note, however, that $1 - e^{-A(z\phi(t))}, z = 0$, cannot be viewed as a Cdf. Therefore, the principle under question implies that: $\lambda_m(t) \sim \psi'(t)$. On the other hand, it follows from (10) that

$$\lambda_m(t) \sim \psi'(t) + (\alpha + 1)(\log \phi(t))'$$

and if the second term on the right hand side of this relation is increasing faster than $\psi'(t)$ as $t \to \infty$, then this term defines asymptotic behavior of $\lambda_m(t)$. It is clear that it is possible for the fast increasing functions (e.g., for $\exp\{t^n\}$, $n \geq 1$). Thus, if then $\psi'(t) = o((\log \phi(t))')$, then

$$\lambda_m(t) \sim (\alpha + 1)(\log \phi(t))',$$

whereas the Principle holds only when $(\log \phi(t))' = o(\psi'(t))$.

4 Specific models

4.1 Multiplicative (PH - proportional hazards) model

In the conventional notation the baseline failure rate is usually denoted as $\lambda_0(t)$ (or $\lambda_b(t)$). Therefore model (4) reads

$$\lambda(t,z) = z\lambda_0(t), \quad \Lambda_0(t) = \int_0^t \lambda_0(u)du \tag{18}$$

and the mixture failure rate is given by

$$\lambda_m(t) = \frac{\int_0^\infty z\lambda_0(t)e^{-z\Lambda_0(t)}\pi(z)dz}{\int_0^\infty e^{-z\Lambda_0(t)}\pi(z)dz}. \tag{19}$$

As $A(u) \equiv u$, $\phi(t) = \Lambda_0(t)$, $\psi(t) \equiv 0$ in this specific case, theorems 1 and 3 are simplified to

Corollary 2 *Assume that the mixing pdf $\pi(z)$, $z \in [0,\infty)$ can be written as*

$$\pi(z) = z^\alpha \pi_1(z), \tag{20}$$

where $\alpha > -1$ and $\pi_1(z)$ is bounded in $[0,\infty)$, continuous at $z=0$ and $\pi_1(0) \neq 0$.

Then the mixture failure rate for the multiplicative model (18) has the following asymptotic behavior:

$$\lambda_m(t) \sim \frac{(\alpha+1)\lambda_0(t)}{\int_0^t \lambda_0(u)du}. \tag{21}$$

Corollary 3 *Assume that the mixing pdf $\pi(z), z \in [a,\infty)$ (we can define $\pi(z) = 0, z \in [0,a)$) is bounded, right semi-continuous at $z=a$ and $\pi(a) \neq 0$.*

Then, in accordance with relation (17), the mixture failure rate for the model (18) has the following asymptotic behavior:

$$\lambda_m(t) \sim a\lambda_0(t) \tag{22}$$

Corollary 2 states a remarkable fact: asymptotic behavior of the mixture failure rate $\lambda_m(t)$ depends only on the behavior of the mixing pdf in the neighborhood of $z=0$ and the baseline failure rate $\lambda_0(t)$.

Corollary 3 describes the convergence of a mixture failure rate to the mixture failure rate of the strongest population. In this simple multiplicative case the family of the failure rates is trivially ordered in z and the strongest population has the failure rate $a\lambda_0(t)$.

The next theorem generalizes the result of Corollary 3:

Theorem 4 *Assume that the mixing pdf* $\pi(z)$ *in model (18) has support in* $[a, b]$, $a < 0, b \leq \infty$, *and for* $z \geq a$ *it can be defined as*

$$\pi(z) = (z - a)^{\alpha}\pi_1(z - a), \tag{23}$$

where $\alpha > -1$, $\pi_1(z)$ *is bounded in* $[0, b - a]$ *and* $\pi_1(0) \neq 0$.

Then

$$\lambda_m(t) \sim a\lambda_0(t). \tag{24}$$

It is quite remarkable, that asymptotic result in this theorem does not depend on a mixing distribution even in the case of a singularity at $z = a$. This differs from the case $a = 0$ in Corollary 2. Relation (24) also describes the convergence to the failure rate of the strongest population, which differs dramatically from the convergence described by (21). Explanation of this difference is quite obvious and due to the multiplicative nature of the model: the behavior of $z\lambda_0(t)$ in the neighborhood of $z = 0$ for the pdf (20) is different from the behavior of this product in the neighborhood of $z = a$ for the pdf (23).

The mixture failure rate given by equation (18) can be obtained explicitly when the Laplace transform of the mixing pdf $\tilde{\pi}(z)$ is easily computed like in Example 2. As the cumulative failure rate is monotonically increasing in t, the mixture survival function is written in terms of the Laplace transform as:

$$\int_0^{\infty} e^{-z\Lambda_0(t)}\pi(z)dz = \tilde{\pi}(\Lambda_0(t)).$$

Therefore, equation turns into

$$\lambda_m(t) = -\frac{(\tilde{\pi}(\Lambda_0(t)))'}{\tilde{\pi}(\Lambda_0(t))} = -(\log\tilde{\pi}(\Lambda_0(t)))'$$

and the corresponding inverse problem can be also solved: given the mixture failure rate and the mixing distribution, obtain the baseline failure rate (Finkelstein and Esaulova, 2001b).

Example 2 Let the mixing distribution be the Gamma distribution with the pdf

$$\pi(z) = \left(\frac{z}{b}\right)^{c-1} e^{-z/b} \frac{1}{b\Gamma(c)}, \tag{25}$$

where $b, c > 0$. The Laplace transform of $\pi(z)$ is $\tilde{\pi}(t) = (tb + 1)^{-c}$ and therefore the mixture failure rate is given by the following expression:

$$\lambda_m(t) = \frac{bc\lambda_0(t)}{1 + b\int_0^t \lambda_0(u)du}. \tag{26}$$

The expected value of a random variable Z with a pdf (25) is bc and the variance is b^2c. Thus for the fixed expectation $E[Z] = 1$ the variance $\sigma^2 = b$ and equation (26) turns into

$$\lambda_m(t) = \frac{\lambda_0(t)}{1 + \sigma^2\int_0^t \lambda_0(u)du}, \tag{27}$$

which first appeared in Vaupel *et al* (1979) in a demographic context. This form al-lows to compare different mixtures for the fixed baseline distribution. We can see that when the variance of the mixing distribution increases, the mixture failure rate decreases.

Obviously, asymptotic behavior of $\lambda_m(t)$ can be explicitly analyzed. Consider two specific cases:

If the baseline distribution is Weibull with $\lambda_0 = \lambda t^\beta$, then the mixture failure rate (26) is (see also Gupta and Gupta, 1996):

$$\lambda_m(t) = \frac{(\beta + 1)\lambda bct^\beta}{(\beta + 1) + \lambda bt^{\beta+1}}, \tag{28}$$

which as $t \to \infty$ converges to 0 and $\lambda_m(t) \sim (\beta+1)ct^{-1}$ exactly as prescribed by our formula (21) of Corollary 2 ($c = \alpha + 1$).

If the baseline distribution is Gompertz with $\lambda_0(t) = \mu e^{\beta t}$, then simple transformations result in

$$\lambda_m(t) = \frac{\beta c e^{\beta t}}{e^{\beta t} + \left(\frac{\beta}{\mu b} - 1\right)}. \tag{29}$$

If $b = \beta/\mu$, then $\lambda_m(t) \equiv \beta c$, if $b > \beta/\mu$, then $\lambda_m(t)$ increases to β/μ, and if $b < \beta/\mu$, it decreases to β/μ.

Coming back to a discussion of convergence of the mixture failure rate to the failure rate of the strongest population in Example 1 of Section 3, it is reasonable to compare asymptotic behavior in (28) and (29) for the same mixing distribution (25). In case of the Weibull Cdf, the mixture failure rate is converging to 0. This means that within the framework of the multiplicative model (18), where the family of failure rates is ordered in z, we still can speak in terms of convergence to the failure rate of the strongest population, defining the case $z = 0$ as some 'generalized' (or formal) strongest failure rate: $\lambda(t, 0) = 0$. As it was mentioned, $1 - e^{-A(z\phi(t))}$ cannot be viewed as a Cdf in this case, which formally violates our general assumptions in Section 2. But the failure rate for a Gompertz Cdf does not converge to 0, it converges to a constant, thus violating the principle of converging to the failure rate of the strongest population even formulated in a 'generalized' form! The reason for that, and this goes in line with our discussion in Example 1, is in the fast increase in the function $\phi(t)$, which is proportional to $e^{\beta t}$ in the latter case.

4.2 Accelerated life model

In a conventional notation this model is written as:

$$\lambda(t, z) = z\lambda_0(tz), \quad \Lambda(t, z) = \Lambda_0(tz) = \int_0^{tz} \lambda_0(u)du \tag{30}$$

Although the definition of the ALM is also very simple, the presence of a mixing parameter z in the arguments make analysis of the mixture failure rate more complex than in the multiplicative case. Therefore, as it was already mentioned, this model was not practically studied before. The mixture failure rate in this specific case is

$$\lambda_m(t) = \frac{\int_0^\infty z\lambda_0(tz)e^{-\Lambda_0(tz)}\pi(z)dz}{\int_0^\infty e^{-\Lambda_0(tz)}\pi(z)dz}. \tag{31}$$

Asymptotic behavior of $\lambda_m(t)$ can be described as a specific case of Theorem 1 with $A(s) = \Lambda_0(s)$, $\phi(t) = t$ and $\psi(t) \equiv 0$:

Corollary 4 *Assume that the mixing pdf $\pi(z), z \in [0, \infty)$ can be defined as $\pi(z) = z^\alpha \pi_1(z)$, where $\alpha > 1$, $\pi_1(z)$ is continuous at $z = 0$ and bounded in $[0, \infty)$, $\pi_1(0) \neq 0$.*

Let the baseline distribution with the cumulative failure rate $\Lambda_0(t)$ have a moment of order $\alpha + 1$. Then

$$\lambda_m(t) \sim \frac{\alpha + 1}{t} \tag{32}$$

as $t \to \infty$.

The conditions of Corollary 4 are not that strong and are relatively natural. The most of the widely used lifetime distributions have all moments. The Pareto distribution will be discussed in the next example.

As it was already stated, the conditions on the mixing distribution hold, e.g., for the Gamma and the Weibull distributions which are commonly used as mixing distributions.

Relation (32) is really surprising, as it does not depend on the baseline distribution, which seems striking at least at the first sight. It is also dramatically different from the multiplicative case (21). It follows from Example 2 that both asymptotic results coincide in the case of the Weibull baseline distribution, which is obvious, as only for the Weibull distribution the ALM can be re-parameterized to end up with a PH model and *vise versa*.

The following example shows other possibilities for the asymptotic behavior of $\lambda_m(t)$ when one of the conditions of the Corollary 4 does not hold.

Example 3 Consider the Gamma mixing distribution $\pi(z) = z^{\alpha}e^{-x}/\Gamma(\alpha + 1)$. Let the baseline distribution be the Pareto distribution with density $f_0(t) = \beta/t^{\beta+1}$, $t \geq 1, \beta > 0$.

For $\beta > \alpha + 1$ the conditions of Corollary 2 holds and relation (32) takes place. Let $\beta \leq \alpha + 1$, which means that the baseline distribution doesn't have the $(\alpha + 1)$th moment. Therefore, one of the conditions of Corollary 4 is violated. In this case it can be shown by direct derivations (see Section 5) that

$$\lambda_m(t) \sim \frac{\beta}{t}$$

as $t \to \infty$, whereas for the general case:

$$\lambda_m(t) \sim \frac{\min(\beta, \alpha + 1)}{t}.$$

It can be shown that the same asymptotic relation holds not only for the Gamma-distribution, but also for any other mixing distribution $\pi(z)$ of the form $\pi(z) = z^{\alpha}\pi_1(z)$. If $\beta > \alpha + 1$, the function $\pi_1(z)$ should be bounded and $\pi_1(0) \neq 0$.

As $A(s) = \Lambda_0(s)$, $\phi(t) = t$, Theorem 3 is simplified to

Corollary 5 *Assume that the mixing pdf $\pi(z)$, $z \in [a, \infty)$ is bounded, continuous at $z = a$ and $\pi(a) \neq 0$. Let*

$$\frac{\lambda_0'(t)}{(\lambda_0(t))^2} \to 0, \quad t\lambda_0(t) \to \infty \tag{33}$$

as $t \to \infty$. Assume also that $\forall\, b, c > 0$, $b < c$ the quotient $\lambda_0(bx)/\lambda_0(cx)$ is bounded as $x \to \infty$. Then, in accordance with relation (17), the mixture failure rate for the model (30) has the following asymptotic behavior:

$$\lambda_m(t) - \psi'(t) \sim a\lambda_0(at). \tag{34}$$

Conditions (33) are rather weak. E.g., the marginal case of the Pareto distribution - the baseline failure rate of the form $\lambda_0(t) = ct^{-1}$, $c > 0, t \geq 1$ does not comply with (33), but in mixing we are primarily interested in increasing, at least ultimately, baseline failure rates.

Asymptotic behavior of $\lambda_m(t)$ in the **additive hazards model** (6) due to its simplicity does not deserve special attention. As $A(s) \equiv s$ and $\phi(t) \equiv t$, conditions (8) and (9) of Theorem 1, for instance, hold and asymptotic result (10) is simplified to:

$$\lambda_m(t) - \psi'(t) \sim \frac{\alpha + 1}{t}.$$

5 Proofs

5.1 Proof of Theorem 1

First we need a simple lemma for Dirac sequence of functions:

Lemma 1 *Let $g(z), h(z)$ be nonnegative locally integrable functions in $[0, \infty)$ satisfying the following conditions:*

$$\int_0^\infty g(z)dz < \infty,$$

and $h(z)$ is bounded and continuous at $z = 0$.

Then, as $t \to \infty$:

$$t \int_0^\infty g(tz)h(z)dz \to h(0) \int_0^\infty g(z)dz. \tag{35}$$

Proof Substituting $u = tz$:

$$t \int_0^\infty g(tz)h(z)dz = \int_0^\infty g(u)h(u/t)du.$$

The function $h(u)$ is bounded and $h(u/t) \to 0$ as $t \to \infty$, thus convergence (35) holds by dominated convergence theorem. $\square$

Now we prove Theorem 1. The proof is straightforward as we use definition (1) and Lemma 1.

The survival function for the model (2) is

$$\bar{F}(t, z) = e^{-A(z\phi(t))-\psi(t)}.$$

Taking into account that $\phi(t) \to \infty$ as $t \to \infty$, and applying Lemma 1 to the function $g(u) = e^{-A(u)}u^\alpha$:

$$\begin{aligned}\int_0^\infty \bar{F}(t, z)\pi(z)dz &= \int_0^\infty e^{-A(z\phi(t))-\psi(t)} z^\alpha \pi_1(z)dz \\ &\sim \frac{e^{-\psi(t)}\pi_1(0)}{\phi(t)^{\alpha+1}} \int_0^\infty e^{-A(s)}s^\alpha ds, \end{aligned} \tag{36}$$

where the integral is finite due to the condition (9). The corresponding probability density function is:

$$\begin{aligned} f(t, z) &= (A'(z\phi(t))z\phi'(t) + \psi'(t))e^{-A(z\phi(t))-\psi(t)} \\ &= A'(z\phi(t))z\phi'(t)e^{-A(z\phi(t))-\psi(t)} + \psi'(t)\bar{F}(t, z). \end{aligned}$$

Similarly, applying Lemma 1:

$$\begin{aligned} &\int_0^\infty f(t, z)\pi(z)dz - \psi'(t) \int_0^\infty \bar{F}(t, z)\pi(z)dz \\ &\qquad = \phi'(t)e^{-\psi(t)} \int_0^\infty A'(z\phi(t))e^{-A(z\phi(t))} z^{\alpha+1}\pi_1(z)dz \\ &\qquad \sim \frac{\phi'(t)e^{-\psi(t)}\pi_1(0)}{\phi(t)^{\alpha+2}} \int_0^\infty A'(s)e^{-A(s)}s^{\alpha+1}ds. \end{aligned} \tag{37}$$

Due to condition (9) and the fact, that $A(s)$ is ultimately increasing,

$$A(s)s^{\alpha+1} \to 0 \quad \text{as } t \to \infty. \tag{38}$$

Indeed, by the mean value theorem

$$\int_s^{2s} e^{-A(u)} u^\alpha du = s e^{-A(s_1)} s_1^\alpha$$

for some $s \le s_1 \le 2s$. The right-hand side tends to 0. For s larger than some s_0 we have $A(s_1) > A(s)$, thus, the left-hand side is smaller than $2^\alpha s^{\alpha+1} e^{-A(s)}$, and this leads to (38). Using it while integrating by parts we get

$$\int_0^\infty A'(s) e^{-A(s)} s^{\alpha+1} ds = (\alpha+1) \int_0^\infty e^{-A(s)} s^\alpha ds. \tag{39}$$

Combining (36)-(39), finally:

$$\frac{\int_0^\infty f(t,z)\pi(z)dz}{\int_0^\infty \bar{F}(t,z)\pi(z)dz} - \psi'(t) \sim (\alpha+1)\frac{\phi'(t)}{\phi(t)}.$$

5.2 Proof of Theorem 2

Lemma 2 *Let $\{g(t,z),\ z \in [0,\infty)\}$ be a family of functions and $h(z)$ a function, satisfying the following conditions:*

(i) for every $z \in [0,\infty)$ the function $g(t,z)$ is integrable in t and for every $t \in [0,\infty)$ it is integrable in z.

(ii) there exists a function $\alpha(t)$, $\alpha(t) \to 0$ as $t \to \infty$ and

$$\frac{\int_0^{\alpha(t)} g(t,z)dz}{\int_0^\infty g(t,z)dz} \to 1 \tag{40}$$

as $t \to \infty$.

(iii) a function $h(x)$ is locally integrable, bounded in $[0,\infty)$ and continuous at $z = 0$.

Then, as $t \to \infty$:

$$\frac{\int_0^\infty g(t,z)h(z)dz}{\int_0^\infty g(t,z)dz} \to h(0).$$

Proof Let $h(z) \leq M$, $z \in [0, \infty)$. Then:

$$\frac{\int_0^\infty g(t,z)h(z)dz}{\int_0^\infty g(t,z)dz} = \frac{\int_0^{\alpha(t)} g(t,z)h(z)dz}{\int_0^\infty g(t,z)dz} + \frac{\int_{\alpha(t)}^\infty g(t,z)h(z)dz}{\int_0^\infty g(t,z)dz}.$$

The second term is majorized by

$$M \frac{\int_{\alpha(t)}^\infty g(t,z)dz}{\int_0^\infty g(t,z)dz}$$

which is due to condition (40). The first term converges to $h(0)$ due to the same condition and the fact that $h(z)$ is continuous at $z = 0$. □

For proving Theorem 2 we first show in a direct way that for $\bar{F}(t,z)$ there holds a condition similar to (11). For every $\varepsilon > 0$ we choose t_ε such that for $u > t_\varepsilon$ the function $\alpha(u)$ already decreases and

$$\int_0^{\alpha(u)} f(u,z)\pi(z)dz > (1-\varepsilon)\int_0^\infty f(u,z)\pi(z)dz.$$

Since $\alpha(t)$ decreases

$$\int_0^{\alpha(t)} f(u,z)\pi(z) > \int_0^{\alpha(u)} f(u,z)\pi(z)dz$$

for $u > t > t_\varepsilon$. Thus

$$\begin{aligned}
\int_0^{\alpha(t)} \bar{F}(t,z)\pi(z)dz &= \int_0^{\alpha(t)} \int_t^\infty f(u,z)du\,\pi(z)dz \\
&= \int_t^\infty \int_0^{\alpha(t)} f(u,z)\pi(z)dz\,du \\
&> \int_t^\infty \int_0^{\alpha(u)} f(u,z)\pi(z)dz\,du \\
&> (1-\varepsilon)\int_t^\infty \int_0^\infty f(u,z)\pi(z)dz\,du \\
&= (1-\varepsilon)\int_0^\infty \bar{F}(t,z)\pi(z)dz.
\end{aligned}$$

Now we apply Lemma 2 with $h(z) = \pi_1(z)/\pi(z)$ and $g(t,z) = f(t,z)\pi(z)$, which results in

$$\frac{\int_0^\infty f(t,z)\rho(z)dz}{\int_0^\infty f(t,z)\rho(z)dz} \to h(0).$$

In a similar way $g(t,z) = \bar{F}(t,z)\pi(z)$ with the same $h(z)$ gives

$$\frac{\int_0^\infty \bar{F}(t,z)\rho(z)dz}{\int_0^\infty \bar{F}(t,z)\rho(z)dz} \to h(0),$$

as $t \to \infty$, and relation (12) follows immediately.

5.3 Proof of Theorem 3

This theorem is rather technical and we must first prove three supplementary lemmas, which present consecutive steps on a way to asymptotic relation (17).

Lemma 3 *Let $h(x)$ be a twice differentiable function with an ultimately positive derivative, and*

$$\int_0^\infty e^{-h(y)}dy < \infty. \tag{41}$$

Let also

$$\frac{h''(x)}{(h'(x))^2} \to 0 \tag{42}$$

as $x \to \infty$. Then

$$\int_x^\infty e^{-h(y)}dy \sim e^{-h(x)}\frac{1}{h'(x)} \tag{43}$$

as $x \to \infty$.

Proof Let x_0 be such that $h'(x) > 0$ for $x > x_0$. Due to (41) $h(x) \to \infty$ as $x \to \infty$. Then there exists an inverse function $g(x)$ defined in $[x_0, \infty)$: $g(h(x)) \equiv h(g(x)) \equiv 1$. The function $g(x)$ is also twice differentiable and $g'(x) = 1/h'(g(x))$. Integrating by parts for $x > x_0$:

$$\begin{aligned}\int_x^\infty e^{-h(y)}dy &= \int_{h(x)}^\infty e^{-u}g'(u)du \\ &= e^{-h(x)}g'(h(x)) + \int_{h(x)}^\infty e^{-u}g''(u)du. \end{aligned} \tag{44}$$

Since

$$\frac{g''(u)}{g'(u)} = -\frac{h''(g(u))}{h'(g(u))^2} \to 0$$

as $u \to \infty$, the right-hand side integral vanishes compared with the one on the left-hand side. Therefore, eventually

$$\int_x^\infty e^{-h(y)} dy \sim e^{-h(x)} g'(h(x)) = e^{-h(x)} \frac{1}{h'(x)} .$$

□

Lemma 4 *Let assumptions of Lemma 3 hold. Assume additionally that as* $x \to \infty$

$$xh'(x) \to \infty \tag{45}$$

and for any $b, c \geq a$, $b < c$ *the quotient* $h'(bx)/h'(cx)$ *is bounded in* $[0, \infty)$.

Let $\mu(u)$ *be a positive, bounded and locally integrable function, defined in* $[a, \infty)$, *continuous at* $u = a$, *and* $\mu(a) \neq 0$.

Then

$$\int_a^\infty e^{-h(ux)} \mu(u) du \sim \frac{\mu(a) e^{-h(ax)}}{xh'(ax)}$$

as $x \to \infty$.

Proof As the first step we prove that:

$$I(x) = \int_a^\infty e^{-h(ux)} \mu(u) du \sim \mu(a) \int_a^\infty e^{-h(ux)} du$$

As $\mu(u)$ is continuous at $u = a$, for $\varepsilon > 0$ there is δ such that $|\mu(u) - \mu(a)| < \varepsilon$, if $|u - a| < \delta$. The function $\mu(u)$ is bounded, therefore, $\mu(u) < M$, $\forall u \in [a, \infty)$ for some positive $M > \mu(a)$. Then

$$I(x) = \int_a^{a+\delta} e^{-h(ux)} \mu(u) du + \int_{a+\delta}^\infty e^{-h(ux)} \mu(u) du$$

and

$$\begin{aligned}
&|I(x) - \mu(a) \int_a^\infty e^{-h(ux)} du| \\
&\qquad < \varepsilon \int_a^{a+\delta} e^{-h(ux)} du + (M - \mu(a)) \int_{a+\delta}^\infty e^{-h(ux)} du \\
&\qquad = \varepsilon \int_a^\infty e^{-h(ux)} du + (M - \mu(a) - \varepsilon) \int_{a+\delta}^\infty e^{-h(ux)} du.
\end{aligned}$$

Then

$$\left|\frac{I(x)}{\mu(a)\int_a^\infty e^{-h(ux)}du} - 1\right| < \frac{\varepsilon}{\mu(a)} + \frac{M-\mu(a)-\varepsilon}{\mu(a)} \cdot \frac{\int_{a+\delta}^\infty e^{-h(ux)}du}{\int_a^\infty e^{-h(ux)}du} \tag{46}$$

Using Lemma 3:

$$\frac{\int_{a+\delta}^\infty e^{-h(ux)}du}{\int_a^\infty e^{-h(ux)}du} = \frac{\int_{ax+\delta x}^\infty e^{-h(u)}du}{\int_{ax}^\infty e^{-h(u)}du} \sim \frac{h'(ax)}{h'(ax+\delta x)} e^{-(h(ax+\delta x)-h(ax))}$$

It follows from the condition (45) and mean value theorem that

$$h(ax+\delta x) - h(ax) = \delta x h'(s) > s h'(s) \frac{\delta}{a+\delta} \tag{47}$$

for some $ax < s < ax + \delta x$. Thus $h(ax+\delta x) - h(ax) \to \infty$ as $x \to \infty$, the quotient $h'(ax)/h'(ax+\delta x)$ is bounded and, therefore, the second summand in (46) tends to zero, whereas the first summand can be made arbitrarily small. This yields

$$I(x) \sim \mu(a) \int_a^\infty e^{-h(ux)}du$$

as $x \to \infty$. Applying Lemma 3 completes the proof. $\square$

Lemma 5 *Under assumptions of Lemma 4 the following asymptotic relation holds as* $x \to \infty$

$$\int_a^\infty h'(ux)e^{-h(ux)} u\mu(u)du \sim \frac{a\mu(a)}{x} e^{-h(ax)}.$$

Proof We first show that

$$\int_a^\infty h'(ux)e^{-h(ux)} u\,du \sim \frac{a}{x}\, e^{-h(ax)}. \tag{48}$$

Simple calculations give

$$\begin{aligned} x^2 \int_a^\infty h'(ux)e^{-h(ux)} u\,du &= \int_{ax}^\infty h'(u)e^{-h(u)} u\,du \\ &= axe^{-h(ax)} + \int_{ax}^\infty e^{-h(u)}du. \end{aligned}$$

By Lemma 4:

$$\int_{ax}^{\infty} e^{-h(u)} du \sim e^{-h(ax)} \frac{1}{h'(ax)}.$$

We have assumed that $axh'(ax) \to \infty$ as $x \to \infty$, thus $\frac{1}{h'(ax)} = o(ax)$ and

$$x^2 \int_0^{\infty} h'(ux)e^{-h(ux)} u \, du \sim axe^{-h(ax)},$$

which is the same as (48).

The next step is to prove that

$$\int_a^{\infty} h'(ux)e^{-h(ux)} u\mu(u) du \sim \mu(a) \int_a^{\infty} h'(ux)e^{-h(ux)} u \, du \tag{49}$$

As in Lemma 4, we use the same ε, δ, M and the similar reasoning to get

$$\left| \frac{\int_a^{\infty} h'(ux)e^{-h(ux)} u\mu(u) du}{\mu(a) \int_a^{\infty} h'(ux)e^{-h(ux)} u \, du} - 1 \right| < \frac{\varepsilon}{\mu(a)} + \frac{\tilde{M}}{\mu(a)} \cdot \frac{\int_{a+\delta}^{\infty} h'(ux)e^{-h(ux)} u \, du}{\int_a^{\infty} h'(ux)e^{-h(ux)} u \, du},$$

where $\tilde{M} = M - \mu(a) - \varepsilon$.

Applying (48) and using (47) we obtain:

$$\frac{\int_{a+\delta}^{\infty} h'(ux)e^{-h(ux)} u \, du}{\int_a^{\infty} h'(ux)e^{-h(ux)} u \, du} \sim \frac{a+\delta}{a} e^{-(h(ax+\delta x) - h(ax))} \to 0$$

as $x \to \infty$. Again $\varepsilon/\mu(a)$ can be made arbitrarily small, which gives us (49). Combining it with (48) completes the proof. □

Now we are ready to prove Theorem 3 itself. Applying Lemma 4 as $t \to \infty$ results in:

$$\begin{aligned}
\int_a^{\infty} \bar{F}(t,z)\pi(z)dz &= \int_a^{\infty} e^{-A(z\phi(t)) - \psi(t)} \pi(z) dz \\
&= e^{-\psi(t)} \int_a^{\infty} e^{-A(z\phi(t))} \pi(z) dz \\
&\sim \frac{\pi(a)e^{-\psi(t)}}{\phi(t)A'(a\phi(t))} e^{-A(a\phi(t))}.
\end{aligned}$$

Similar to the proof of Theorem 1:

$$\int_a^\infty f(t,z)\pi(z)dz - \psi'(t)\int_a^\infty \bar{F}(t,z)\pi(z)dz$$
$$= \phi'(t)e^{-\psi(t)}\int_a^\infty A'(z\phi(t))e^{-A(z\phi(t))}z\pi(z)dz.$$

Using Lemma 5:

$$\int_a^\infty A'(z\phi(t))e^{-A(z\phi(t))}z\pi(z)dz \sim \frac{a\pi(a)}{\phi(t)}e^{-A(a\phi(t))}.$$

Combining the last three statements arrive at (17)

$$\begin{aligned}\lambda_m(t) - \psi'(t) &= \frac{\int_a^\infty f(t,z)\pi(z)dz}{\int_a^\infty \bar{F}(t,z)\pi(z)dz} - \psi'(t)\\ &= \frac{\phi'(t)e^{-\psi(t)}a\pi(a)e^{-A(a\phi(t))}}{\phi(t)} \cdot \frac{A'(a\phi(t))\phi(t)}{\pi(a)e^{-\psi(t)}e^{-A(a\phi(t))}}\\ &= a\phi'(t)A'(a\phi(t)).\end{aligned}$$

5.4 Proof of Theorem 4

As in Theorem 1, we consider the numerator and the denominator in (19) separately. Changing the variables and applying Lemma 1:

$$\begin{aligned}\int_0^\infty \bar{F}(tz)\pi(z)dz &= \int_a^\infty e^{-z\Lambda_0(t)}(z-a)^\alpha\pi_1(z-a)dz\\ &= e^{-a\Lambda_0(t)}\int_0^\infty e^{-z\Lambda_0(t)}z^\alpha\pi_1(z)dz\\ &\sim \frac{e^{-a\Lambda_0(t)}\pi_1(0)\Gamma(\alpha+1)}{(\Lambda_0(t))^{\alpha+1}}.\end{aligned} \tag{50}$$

Similarly,

$$\begin{aligned}\int_0^\infty zf(tz)\pi(z)dz &= \lambda_0(t)\int_a^\infty ze^{-z\Lambda_0(t)}(z-a)^\alpha\pi_1(z-a)dz\\ &= \lambda_0(t)e^{-a\Lambda_0(t)}\int_0^\infty e^{-z\Lambda_0(t)}z^{\alpha+1}\pi_1(z)dz\\ &\quad + a\lambda_0(t)e^{-a\Lambda_0(t)}\int_0^\infty e^{-z\Lambda_0(t)}z^\alpha\pi_1(z)dz.\end{aligned}$$

The first integral on the right hand side as $t \to \infty$ is equivalent to $\pi_1(0)\Gamma(\alpha+2)(\Lambda_0(t))^{-\alpha-2}$ and the second to $\pi_1(0)\Gamma(\alpha+1)(\Lambda_0(t))^{-alpha-1}$, which decreases slower. Thus

$$\int_0^\infty zf(tz)\pi(z)dz \sim a\pi_1(0)\Gamma(\alpha+1)\lambda_0(t)\frac{e^{-a\Lambda_0(t)}}{(\Lambda_0(t))^{\alpha+1}}\,. \tag{51}$$

Finally using (50) and (51) in (19), we arrive at (24).

5.5 Proof of the Example 3

Calculating directly:

$$\begin{aligned}\int_0^\infty f_0(tz)z\pi(z)dz &= \int_{1/t}^\infty \frac{\beta z}{t^{\beta+1}z^{\beta+1}}\cdot\frac{1}{\Gamma(\alpha+1)}e^{-z}z^\alpha dz\\ &= \frac{\beta}{\Gamma(\alpha+1)t^{\beta+1}}\int_{1/t}^\infty z^{\alpha-\beta}e^{-z}dz\\ &\sim \frac{\Gamma(\alpha-\beta+1)\beta}{\Gamma(\alpha+1)t^{\beta+1}}\end{aligned}$$

and

$$\int_0^\infty \bar{F}_0(tz)\pi(z)dz = \int_0^{1/t}\frac{e^{-z}z^\alpha}{\Gamma(\alpha+1)}dz + \int_{1/t}^\infty \frac{1}{t^\beta z^\beta}\cdot\frac{e^{-z}z^\alpha}{\Gamma(\alpha+1)}dz,$$

As $t \to \infty$ the first integral on the right-hand side is equivalent to

$$\frac{1}{\Gamma(\alpha+1)}\int_0^{1/t} z^\alpha dz = \frac{1}{t^{\alpha+1}\Gamma(\alpha+2)}$$

and the second integral is equivalent to $\Gamma(\alpha-\beta+1)/\Gamma(\alpha+1)t^\beta$, which in case $\beta \le \alpha$ decreases slower; therefore the sum of two integrals is equivalent to $\Gamma(\alpha-\beta+1)/\Gamma(\alpha+1)t^\beta$.

Eventually

$$\lambda_m(t) \sim \frac{\Gamma(\alpha-\beta+1)\beta}{\Gamma(\alpha+1)t^{\beta+1}}\cdot\frac{\Gamma(\alpha+1)t^\beta}{\Gamma(\alpha-\beta+1)} = \frac{\beta}{t}.$$

If $\beta = \alpha+1$, then

$$\int_0^\infty zf_0(tz)\pi(z)dz = \frac{\alpha+1}{\Gamma(\alpha+1)t^{\alpha+2}}\int_{1/t}^\infty z^{-1}e^{-z}dz,$$

and since

$$\int_0^{1/t} z^\alpha dz = o\left(t^{-\alpha-1}\int_{1/t}^\infty z^{-1}e^{-z}dz\right)$$

we obtain:

$$\begin{aligned}\int_0^\infty \bar{F}_0(tz)\pi(z)dz &\sim \int_{1/t}^\infty \frac{1}{t^{\alpha+1}z}\cdot\frac{e^{-z}}{\Gamma(\alpha+1)}dz\\ &= \frac{1}{t^{\alpha+1}\Gamma(\alpha+1)}\int_{1/t}^\infty z^{-1}e^{-z}dz.\end{aligned}$$

Therefore

$$\lambda_m(t) \sim \frac{\alpha+1}{t} = \frac{\beta}{t}.$$

6 Concluding remarks

Two types of results on the mixture failure rates modelling were primarily considered in the literature. On one hand, general asymptotic results of Clarotti and Spizzichino (1990), Block and Joe (1997) and Block *et al* (1993), where under rather stringent conditions a general asymptotic behavior of the mixture failure rate was studied, on the other hand, specific proportional (additive) hazards-type models of Gurland and Sethuraman (1995), Lynn and Singpurwalla, (1997), Finkelstein and Esaulova (2001a), to name a few, where some more detailed convergence properties were described. It is worth noting, however, that asymptotic behavior of the mixture failure rate for the accelerated life model was not studied before, as approaches used for portional hazards and additive hazards models, did not work in that case.

The survival model (2) of this paper generalizes all three conventional models and creates possibility of deriving explicit asymptotic results. Theorem 1, e.g., defines asymptotic mixture failure rate for the case when the mixing variable is defined in $[0,\infty)$, whereas Theorem 3 does so for $[a,\infty)$, $a>0$.

Some of the obtained results can be generalized to a wider than (2) class of life-time distributions, but it looks like that the considered class is, in a way, 'optimal' in terms of the trade-off between the complexity of a model and tractability (or applicability) of results.

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